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N° 2984
Septembre 1996

THÈME 4

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A FAST COMPUTATIONAL PROCEDURE TO SOLVE THE MULTI-ITEM SINGLE MACHINE LOT SCHEDULING OPTIMIZATION PROBLEM. THE AVERAGE COST CASE

UNE PROCEDURE NUMERIQUE POUR L'OPTIMISATION DE LA PRODUCTION D'UNE MACHINE MULTI-PRODUIT

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Abstract

In this paper we present some especial procedures for the numerical solution of the optimal scheduling problem of a multi-item single machine. We study the infinite horizon case and the optimization criterion is the average cost. We establish the solution of the problem in terms of viscosity solutions of the Quasi-Variational Inequality (QVI) system associated to the problem. The existence of solution of the QVI and the uniqueness of the optimal average cost are proved. A method of discretization and a computational procedure are described which allows us to compute the solution in a short time and with precision of order k. We obtain an estimate for the discretization error and develop an algorithm that converges in a finite number of steps. In our method the nodes of the triangulation mesh are joined by segments of trajectories of the original system. This feature allows us to obtain the k-order precision which, in general, is impossible to obtain by usual methods.

Résumé

On présente ici quelques procédés spéciaux pour la solution numérique du problème optimal d'une machine multi-produit pour le cas d'horizon infini quand le critère d'optimisation est le coût moyen. On établit la solution du problème comme la solution de viscosité d'un système d'inéquations quasi-variationnelles (QVI) associées au problème. On montre l'existence de solution du système QVI et l'unicité du coût optimal moyen. On donne une estimation pour l'erreur de discrétisation et on développe aussi un algorithme très efficient qui converge dans un nombre fini de pas. On fait la description d'une méthode de discrétisation et d'un procédé permettant d'obtenir la solution en employant des petits temps de calcul avec un précision d'ordre k, k étant la mesure de la discrétisation. La caractéristique principale de cette méthode est le fait que les nodes de la triangulation sont unis par des segments des trajectoires du système originel. Cette caractéristique permet d'obtenir la précision d'ordre k qui, en général, est impossible d'obtenir par les méthodes habituelles.

Keywords: *scheduling problems, quasi-variational inequalities, Bellman equation, viscosity solution, average cost, numerical solution.*

Mots clefs: *problèmes d'ordonnacement, inéquations quasi-variationnelles, équation de Bellman, solution de viscosité, coût moyen, solution numérique.*

AMS Classification: Primary: 90B30, 90B35, 90B05, 90B10
Secondary: 90C39, 49B10, 65K10

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1 Introduction

In this paper we study the optimization of the production schedule of a multi-item single machine (see [2], [12], [13]). The objective is to find an optimal production schedule that minimizes the average cost for an infinite horizon. Speciffically we will try to minimize the following criterion

$$J(\alpha(\cdot)) = \limsup_{\nu \to \infty} \frac{1}{\theta_\nu} \sum_{i=1}^{\nu} \left(\int_{\theta_{\nu-1}}^{\theta_\nu} f(y(s), d_{i-1})\, ds + q(d_{i-1}, d_i) \right), \tag{1}$$

where θ_i are the switching times of the control policies used (see [1], [4], [8], [22], [24], [25], [26], [27], [28] and [29] for a more general description of similar problems).
By using dynamic programming techniques (see [9]) and taking into account the switching cost, it is possible to find an optimal feedback policy, in terms of any solution in the viscosity sense of the following first order Quasi-Variational Inequalities (QVI) system.

$$\left| \begin{array}{ll} \dfrac{\partial U_d}{\partial x} g(d) + f - \mu \geq 0 & \text{in } \Omega, \\ U_d \leq S^d(U) & \text{in } \Omega, \\ \left(\dfrac{\partial U_d}{\partial x} g(d) + f - \mu \right) \left(U_d - S^d(U) \right) = 0 & \text{in } \Omega, \end{array} \right. \tag{2}$$

being

$$S^d(U)(x) = \min_{x \neq \hat{x}} \left\{ q(d, \tilde{d}) + U_{\tilde{d}}(x) \right\} \qquad x \in Q,\ d \in D. \tag{3}$$

This system is obtained considering a sequence of optimization problems with non zero discount rate. In a strict sense, the relation between the discount problem and the optimization with average cost problem is: $\forall x \in Q$, $\forall\, d \in D$

$$\lim_{\lambda \to 0} \lambda U_\lambda = \mu = \inf_{\alpha(\cdot)} J(\alpha(\cdot)).$$

To obtain the numerical solution of system (2), we develop in this work a procedure based fundamentally in this convergence property, whose validity holds for the continuous problem as well as for the formulation associated to the fully discrete problem. We will obtain an estimation of the following type

$$|\mu - \mu_k| \leq C\, k.$$

We propose an algorithm which converges in a finite number of steps. We also present numerical results for the case $m = 2$ (optimization of a machine with two items).

2 Description of the problem

2.1 Description of the production system

At any time the machine is either idle or producing any one of m different items. We will denote by

- $d = 0$ the idle state of the machine
- $d = 1, ..., m$, when it is producing item d.

For each item $d = 1, ..., m$; we define the problem data as follows

- r_d the demand by unit time of item d
- p_d the production quantity by unit time at the machine setting d
- M_d the inventory capacity constraint of item d
- $q(d, \tilde{d})$ the switching cost of the machine from state d to $\tilde{d}$
- $f(x, d)$ the instantaneous inventory-holding/production cost

We will always assume a *non zero loop cost condition*:

$\exists q_0 > 0$ such that for any closed loop $d_0, d_1, ..., d_p, d_{p-1}$, with $d_0 = d_{p-1}$, $p \leq m$, we have

$$\sum_{i=0}^{p} q(d_i, d_{i-1}) \geq q_0 \tag{4}$$

and we suppose that the following conditions are verified

$$\begin{aligned} &q(d, \tilde{d}) \geq 0 \qquad \forall \tilde{d} \neq d. \qquad q(d, d) = 0 \qquad \forall d \in D, \\ &q(d, \bar{d}) \leq q(d, \tilde{d}) + q(\tilde{d}, \bar{d}), \qquad \forall d \neq \tilde{d} \neq \bar{d}. \end{aligned} \tag{5}$$

In addition, we assume that the switching time is small enough to be disregarded and that the following condition, under which a feasible schedule exist, holds

$$\sum_{d=1}^{m} \frac{r_d}{p_d} < 1. \tag{6}$$

In fact, we will always assume (6), because condition $\sum_{d=1}^{m} \frac{r_d}{p_d} = 1$, forbids the machine to be in the idle state except for a total time $\tau = \sum_{d=1}^{m} \frac{x_d}{p_d}$, and this is not a natural condition for a problem with infinite horizon.

2.2 The set Q of admissible states

Let $y_d(t)$ be the inventory level of item d at time t, starting at $y_d(0) = x_d$. Therefore, for the global state "y" of the system, we have

$$\begin{aligned} y(t) &= (y_1(t), ..., y_m(t)) \\ (y_1(0), ..., y_m(0)) &= (x_1, ..., x_m). \end{aligned} \tag{7}$$

As neither backlogging nor production over the capacity constraints are allowed for the inventory state y_d, the following restriction holds

$$0 \leq y_d \leq M_d, \qquad \forall d = 1, ..., m. \tag{8}$$

Let us divide the x_i values into three categories

$$\left| \begin{array}{l} x_i = 0\,, \\ 0 < x_i < M_i\,, \\ x_i = M_i\,. \end{array} \right. \tag{9}$$

An x point is classified using an m-tuple of digits $a(x) = (a_1, ..., a_m)$, where

$$\left| \begin{array}{lcl} x_i = 0 & \Rightarrow & a_i = 0\,, \\ 0 < x_i < M_i & \Rightarrow & a_i = 1\,, \\ x_i = M_i. & \Rightarrow & a_i = 2\,. \end{array} \right. \tag{10}$$

Let us define

$$\Gamma(a_1, ..., a_m) \equiv \{x \ : \ a(x) = (a_1, ..., a_m)\}\,.$$

The set Q of admissible states comprises only the set of points with at most one zero component, because if we start from other points that do not verify this condition, we cannot avoid the shortage of at least one item, i.e.

$$Q = \bigcup_a \{\Gamma(a_1, ..., a_i, ..., a_m) \ : \ \text{at most one component } a_i = 0\}\,. \tag{11}$$

Let us denote with ∂Q^- the points of Q that are not admissible, i.e.

$$\partial Q^- = \bigcup_a \{\Gamma(a_1, ..., a_i, ..., a_m) \ : \ \text{at least two } a_i = 0\}\,.$$

If we denote with Ω the interior of Q, we have

$$\Omega \equiv \{x \ : \ 0 < x_i < M_i\,, \ i = 1, ..., m\} = \Gamma(1, ..., 1). \tag{12}$$

2.3 The evolution of the system

For any step function $\alpha : [0, \infty) \to D$ from the definition of r_d, p_d, the following equation of evolution holds

$$\frac{dy}{dt} = g(\alpha(t)), \tag{13}$$

where

$$g(\alpha) = (g_1(\alpha), ..., g_m(\alpha)), \tag{14}$$

being

$$g_d(\alpha) = \left| \begin{array}{ll} -r_d & \text{if } \alpha \neq d, \\ & \\ p_d - r_d & \text{if } \alpha = d. \end{array} \right.$$

Remark 2.1 Since g is piece-wise constant, the equation (13) has global solution for any control policy. At the same time we always suppose that the function f is uniformly Lipschitz in Q, $\forall d \in D$.

2.4 The set A_x^d of admissible controls

An admissible schedule is characterized by a sequence of pairs $\{\theta_i, d_i\}$, where θ_i is the switching time,

$$0 \leq \theta_0 \leq \theta_1 < ... < \theta_i < \theta_{i+1} < ... \tag{15}$$

and $d_i \in D$; $d_i \neq d_{i-1}$; $i = 0, 1, ...$ is the state of production in $(\theta_i, \theta_{i+1}]$.
For each $x \in Q$, $d \in D$, we denote A_x^d the set of all admissible schedules with initial state x and initial machine setting d

$$A_x^d = \left\{ \alpha(\cdot) = (\theta_i, d_i)_{i=0}^{\infty} \ : \ d_0 = d, \ \forall t \in \Re^+, \ y(t) \in Q \right\}. \tag{16}$$

In other words, we will consider sequences $\{\theta_i, d_i\}$ such that the associated trajectories remain in Q, $\forall t \geq 0$.

2.5 The average cost

To each control policy $\alpha(\cdot)$ we associate the cost function (1). For each $d \in D$ and $x \in Q$, we define the minimum average cost

$$\mu_d(x) = \inf \left\{ J(\alpha(\cdot)) \ : \ \alpha(\cdot) \in A_x^d \right\}. \tag{17}$$

Our objective is to find $\forall x \in Q$ and $\forall d \in D$, a policy $\bar{\alpha}_x^d(\cdot) \in A_x^d$, such that

$$J(\bar{\alpha}_x^d(\cdot)) = \mu_d(x). \tag{18}$$

In the following proposition we will see that $\mu_d(x)$ does not depend of d and x.

Proposition 2.1 $\exists \mu \in \Re$ *such that*

$$\mu_d(x) = \mu \qquad \forall x \in Q, \ \forall d \in D. \tag{19}$$

Proof. Let $\hat{x}, x \in Q$ and $\hat{d}\, d \in D$. Taking into account the hypotheses about the dynamic of the system, it is immediate to prove, by using the same techniques as those employed in [13], that there exists a policy $\alpha(\cdot) \in A_x^d$ and $\hat{t} \geq 0$, such that

$$\alpha(\hat{t}) = \hat{x}, \qquad y_x(\hat{t}) = \hat{d}.$$

Let $\hat{\alpha} \in A_{\hat{x}}^{\hat{d}}$, we define $\alpha_1 \in A_x^d$ such that

$$\alpha_1(t) = \left| \begin{array}{ll} \alpha(t) & 0 \leq t \leq \hat{t}, \\ \\ \hat{\alpha}(t - \hat{t}) & t > \hat{t}. \end{array} \right.$$

In consequence

$$J(\alpha_1(\cdot)) = J(\hat{\alpha}(\cdot)).$$

From the definition (17)

$$\mu_d(x) \leq J(\alpha_1(\cdot)),$$

then, as the policy $\hat{\alpha}(\cdot)$ is an arbitrary admissible one, it holds

$$\mu_d(x) \leq \mu_{\hat{d}}(\hat{x}).$$

As (x, d) and $\left(\hat{x}, \hat{d}\right)$ are also arbitrary, we conclude

$$\mu_{\hat{d}}(x) = \mu_d(x) = \mu.$$

□

3 Use of viscosity techniques

We will use viscosity techniques (see [7]), in order to consider general solutions of the Hamilton-Jacobi-Belman (H-J-B) equations system associated to this problem. Also with these techniques, we can easily prove properties of uniqueness of solution and convergence results.

3.1 Definition of the QVI system and its viscosity solution

Let us define

$$\partial Q_e = \bigcup_{i=1}^{m} \left(\gamma_i^+ \bigcup \gamma_i^-\right), \tag{20}$$

where

$$\begin{aligned} \gamma_d^+ &= \bigcup_a \{\Gamma(a_1, ..., a_d, ..., a_m) \ : \ a_d = 2\} \bigcap Q, \\ \gamma_d^- &= \bigcup_a \{\Gamma(a_1, ..., a_d, ..., a_m) \ : \ a_d = 0\} \bigcap Q, \end{aligned} \tag{21}$$

and

$$\partial Q_e = \bigcup_a \{\Gamma(a_1, ..., a_d, ..., a_m) \ : \ \text{at least two coefficients } a_d = 0\} . \tag{22}$$

By using the same methodology as that employed in [15], [16], we will say that, $(U_d\,, \mu)$ are a viscosity solution of (2), with boundary conditions (23)-(25) if

- they are continuous functions in Q
- they satisfy $U_d \leq S^d(U)$
- they verify the following boundary conditions, for $x \in (\partial Q_e \bigcup \partial Q^+)$

$$U_{\tilde{d}}(x) = S^{\tilde{d}}(U)(x) \qquad \forall \tilde{d} \neq d \quad \text{if } x \in \gamma_{\tilde{d}}^- \tag{23}$$

$$U_d(x) = S^d(U)(x) \qquad\qquad \text{if } x \in \gamma_d^- \tag{24}$$

$$\lim_{x \to \partial Q^+} U_d(x) = +\infty \tag{25}$$

- $\dfrac{\partial U_d}{\partial x} g\,(d) + f - \mu \geq 0$ in Ω in the viscosity sense, i.e.

  $\forall \psi \in C^1(\Omega)$, if $U_d - \psi$ has a local maximum in x_0, then

$$\frac{\partial \psi}{\partial x}(x_0)\, g(d) + f(x_0, d) - \mu \geq 0$$

- $\forall x \in \Omega \ / \ U_d(x) < S^d(U)(x)$, then $\exists \delta\,(x)$ such that

$$\frac{\partial U_d}{\partial x} g\,(d) + f - \mu = 0 \qquad \text{in } B_{\delta(x)} \text{ in the viscosity sense, i.e.}$$

$\forall \psi \in C^1(B_{\delta(x)}(x))$, if $U_d - \psi$ has a local maximum in x_0, then

$$\frac{\partial \psi}{\partial x}(x_0)\, g(d) + f(x_0, d) - \mu \geq 0,$$

$\forall \psi \in C^1(B_{\delta(x)}(x))$, if $U_d - \psi$ has a local minimum in x_0, then

$$\frac{\partial \psi}{\partial x}(x_0)\, g(d) + f(x_0, d) - \mu \leq 0.$$

3.2 Uniqueness property in terms of viscosity

A constructive procedure for an optimal policy

Let U_d be any set of continuous functions, which are solutions in the viscosity sense of the QVI system (2), with boundary conditions (23)-(25). By using them, an optimal feedback policy $\alpha^* = \{\theta_i, d_i\} \in A_x^d$ can be obtained in the following way:

We define

$$\theta_0 = 0, \qquad d_0 = d\,,$$

and recursively

$$\theta_i = \min\left\{t \geq \theta_{i-1} \ : \ U_{d_{i-1}}(y(t)) = \left(S^{d_{i-1}}(U)\right)(y(t))\right\}, \tag{26}$$

$$d_i \in \left\{d \in D \ : \ \left(S^{d_{i-1}}(U)\right)(y(\theta_i)) = q(d_{i-1}, d_i) + U_d(y(\theta_i)),\ d_{i-1} \neq d_i\right\}. \tag{27}$$

Next theorem establishes the optimality of the procedure.

Theorema 3.1 *If U is a continuous viscosity solution of the system (2), (23)-(25), then the policy constructed according to (26)-(27), satisfies*

$$J(\alpha^*(\cdot)) = \inf\left\{J(\alpha(\cdot)) \ : \ \alpha \in A_x^d\right\}.$$

The proof uses essentially the procedure employed in [13], [16] and it will not be included here for the sake of briefness.

Corolary 3.1 *There exists at most one value of the parameter μ such that (2), (23)-(25) has a solution in the viscosity sense.*

Proof. Let us suppose that (U^1, μ_1), and (U^2, μ_2) are two solutions in the sense of viscosity of (2), (23)-(25).
It can be seen, by using the same techniques employed in [16], that in terms of a solution U^i we can find $\bar{\alpha}_i(\cdot)$ such that

$$\mu_i = J(\bar{\alpha}_i(\cdot))$$

and $\forall \alpha \in A_x^d$, it is verified

$$J(\alpha(\cdot)) \geq \mu_i\,.$$

In consequence

$$\mu_1 \leq J(\bar{\alpha}_2(\cdot)) = \mu_2 \leq J(\bar{\alpha}_1(\cdot)) = \mu_1\,.$$

□

Remark 3.1 If U is a solution of (2), (23)-(25), then $U + c \cdot e$, is also a solution $\forall c \in \Re$, being $e = (1, ..., 1) \in \Re^{m+1}$.

3.3 Existence of viscosity solution

3.3.1 The discount problem

A Lipschitz-continuous solution of the system (2) with boundary conditions (23)-(25) can be obtained considering a sequence of optimization problems with non zero discount rate (we denote λ this coefficient). For this type of problem, the solution is given by the unique solution in the viscosity sense of the (QVI) system (28), with the boundary conditions (23)-(25).

$$\left|\begin{array}{ll} \dfrac{\partial U_d^\lambda}{\partial x} g(d) + f - \lambda U_d^\lambda \geq 0 & \text{in } \Omega, \\[2ex] U_d^\lambda \leq S^d(U^\lambda) & \text{in } \Omega, \\[2ex] \left(\dfrac{\partial U_d^\lambda}{\partial x} g(d) + f - \lambda U_d^\lambda\right)(U_d^\lambda - S^d(U^\lambda)) = 0 & \text{in } \Omega. \end{array}\right. \tag{28}$$

The viscosity solution is defined in the same form as that used for the system (2), (23)-(25), (see [2], [15], [16]).

3.3.2 Relation between the two problems

The relation between the discount problem and the optimal average problem is given by the following theorem. It gives a strict statement of the intuitive fact that problems with a low discount rate or with average cost are optimized by similar policies.

Theorema 3.2 *By virtue of the feasibility condition* (6), *the following properties hold*

$$\begin{array}{ll} \lim\limits_{\lambda\to 0} \lambda U_d^\lambda = \mu & \forall x \in Q,\ \forall d \in D \\[2ex] \forall U \in \left(\bigcap\limits_{\varsigma>0}\left(\overline{\bigcup\limits_{\varsigma>\lambda>0} (U^\lambda(\cdot) - U_{d_0}^\lambda(x^0)\cdot e)}\right)\right), & \\[2ex] (U,\mu) \text{ is solution of } (2),\ (23)-(25). & \end{array} \tag{29}$$

Remark 3.2 In (29) and in the following proof we consider the topology of uniform convergence in C(Q) .

Proof. As we have seen in the previous section, the following estimation holds

$$0 \leq U_d^\lambda \leq C\left(1 + \frac{1}{\lambda} + (\log(d(x,\partial Q^-)))^-\right), \qquad \forall x \in Q,\ \forall d \in D,$$

in consequence for $x \in Q,\ d \in D$, it results

$$\left|\lambda U_d^\lambda(x)\right| \leq C\left(1 + \lambda\left(1 + (\log(d(x,\partial Q^-)))^-\right)\right). \tag{30}$$

Moreover, when K is a compact subset of Q, the functions U^λ are uniformly Lipschitz continuous, being the Lipschitz constant $L(K)$ independent of λ, i.e. we have

$$\|U_d^\lambda(\cdot) - U_{d_0}^\lambda(x^0)\|_{W^{1,\infty}(K)} \leq L(K). \tag{31}$$

Let

$$U \in \left(\bigcap_{\varsigma>0}\left(\overline{\bigcup_{\varsigma>\lambda>0}(U^\lambda(\cdot) - U_{d_0}^\lambda(x^0)\cdot e)}\right)\right),$$

$$\sigma \in \left(\bigcap_{\varsigma>0}\left(\overline{\bigcup_{\varsigma>\lambda>0}\lambda U_{d_0}^\lambda(x^0)}\right)\right).$$

Then, there exists a sub-sequence $\lambda_\nu \to 0$ (that, in order to alleviate the notation, we will denote λ), with the following properties

$$U_d^\lambda(\cdot) - U_{d_0}^\lambda(x^0) \to U_d(\cdot). \tag{32}$$

$$\lambda U_{d_0}^\lambda(x^0) \to \sigma. \tag{33}$$

We should remark that (31) implies

$$\lambda U_d^\lambda(x) \to \sigma \qquad \forall (x,d) \in Q \times D.$$

Let us see that (U,σ) is a solution in the viscosity sense of (2), (23)-(25). By virtue of the uniform convergence in (32), the following properties are verified

- U_d are continuous functions in Q
- U_d satisfies $U_d \leq S^d(U)$
- U_d satisfies (23)-(25)
- $\dfrac{\partial U_d}{\partial x} g(d) + f - \sigma \geq 0$ in Ω in the viscosity sense; in fact,

  let $\psi \in C^1(\Omega)$ such that

  $$(U_d - \psi)(\tilde{x}) = 0 > (U_d - \psi)(x) \qquad \forall x \in N(\tilde{x})\backslash\{\tilde{x}\}$$

  (where we denote $N(\tilde{x})$ a neighborhood of point $\tilde{x} \in \Omega$).
  For each λ, let $x^\lambda \in N(\tilde{x})$ such that it is a local maximum of $(U_d - \psi)$, i.e.

  $$(U_d^\lambda - \psi)(x^\lambda) \geq (U_d^\lambda - \psi)(x) \qquad \forall x \in N(\tilde{x}).$$

  Since $\tilde{x}$ is a strict maximum of $U_d - \psi$ in $N(\tilde{x})$, and as we are working with a uniformly convergent sub-sequence of $\{U^\lambda(\cdot) - U_{d_0}^\lambda(x^0)\}$, we have

  $$x^\lambda \to \tilde{x}.$$

Then, as U^λ is a viscosity solution of (28), it holds

$$\frac{\partial \psi}{\partial x}(x^\lambda)\, g(d) + f(x^\lambda, d) - \lambda\, U_d^\lambda(x^\lambda) \geq 0$$

by taking limit as $\lambda \to 0$, it results

$$\frac{\partial \psi}{\partial x}(\tilde{x})\, g(d) + f(\tilde{x}, d) - \sigma \geq 0.$$

- $\forall \tilde{x} \in \Omega \;/\; U_d(\tilde{x}) \leq S^d(U)(\tilde{x})$, then $\exists \delta(\tilde{x})$ such that

$$\frac{\partial U_d}{\partial x}\, g(d) + f - \sigma = 0 \qquad \text{in } B_{\delta(\tilde{x})}(\tilde{x}) \text{ in the viscosity sense.}$$

Let $\psi \in C^1(B_{\delta(\tilde{x})}(\tilde{x}))$, such that

$$(U_d - \psi)(\tilde{x}) > (U_d - \psi)(x) \qquad \forall x \in B_{\delta(\tilde{x})}(\tilde{x}) \backslash \{\tilde{x}\}.$$

In a similar way it is proved

$$\frac{\partial \psi}{\partial x}(\tilde{x})\, g(d) + f(\tilde{x}, d) - \sigma \geq 0.$$

To obtain the other inequality, let $\psi \in C^1(B_{\delta(\tilde{x})}(\tilde{x}))$, such that

$$(U_d - \psi)(\tilde{x}) < (U_d - \psi)(x) \qquad \forall x \in B_{\delta(\tilde{x})}(\tilde{x}) \backslash \{\tilde{x}\}.$$

As a consequence of both the uniform convergence of $\{U^\lambda(\cdot) - U_{d_0}^\lambda(x^0)\}$ and the continuity of operator S, for λ large enough it results

$$U_d^\lambda \leq S^d(U^\lambda) \qquad \text{in } B_{\delta(\tilde{x})}(\tilde{x}).$$

For each λ let $x^\lambda \in B_{\delta(\tilde{x})}(\tilde{x})$, such that

$$(U_d^\lambda - \psi)(x^\lambda) \leq (U_d^\lambda - \psi)(x) \qquad \forall x \in B_{\delta(\tilde{x})}(\tilde{x}).$$

Then, by definition of viscosity solution of the system (28), it results

$$\frac{\partial \psi}{\partial x}(x^\lambda)\, g(d) + f(x^\lambda, d) - \lambda\, U_d^\lambda(x^\lambda) \leq 0.$$

Clearly, as $x^\lambda \to \tilde{x}$, by taking limit as $\lambda \to 0$, it results

$$\frac{\partial \psi}{\partial x}(\tilde{x})\, g(d) + f(\tilde{x}, d) - \sigma \leq 0.$$

Then (U, σ) is a solution in the sense of viscosity of (2). By virtue of proposition 2.1, it results $\sigma = \mu$ and in consequence every sequence λU^λ converges to μ.

□

4 The discrete problem

4.1 Elements of the discrete problem

To define the discrete problem, we introduce an approximation which comprises a discretization of the space $W^{1,\infty}_{loc}(\Omega)$ and a discretization of conditions (23)-(24). We use the techniques analyzed in [3], [5], [14], [18], [19], [20] and the notations employed in [2].

4.1.1 Approximation of domain Q

We will approximate Q with $Q_k = \bigcup_j S_j^k$, where S_j^k is a finite set of quadrilateral elements and, in consequence, Q_k is a polyhedron of $\Re^m$. We define

$$k = \max_j(\text{diam}(S_j^k)).$$

We use a special uniform mesh B^k of the space $\Re^m$. This mesh is defined in terms of an arbitrary parameter h, in the following way

$$\begin{aligned}
B^k &= \left\{ x^0 + \sum_{d=0}^{m} \varsigma_d \; e^d \;\; : \;\; \varsigma_d \;\; \text{integer} \right\} \qquad (34)\\
h_d &= \frac{r_d}{p_d} h \\
h_0 &= \left(1 - \sum_{d=1}^{m} \frac{r_d}{p_d} \right) h \\
e^0 &= (-r_1, ..., -r_i, ..., -r_m) \, h_0 \\
e^d &= (-r_1, ..., -r_{d-1}, \, p_d - r_d, -r_{d-1}, ..., -r_m) \, h_d
\end{aligned}$$

We will say that S_j^k is an elementary domain of Q_k if it has the following form

$$S_j^k = x^k + \left\{ x = \sum_{d=0}^{m} \varsigma_d \; e^d \;\; : \;\; \varsigma_d \in [0,1] \right\}, \qquad x^k \in B^k, \qquad S_j^k \subset Q. \qquad (35)$$

We will denote with $V^k = \{x^i \, , \, i = 1, ...N\}$ the set of nodes of Q_k and we will denote the cardinal of V^k by N. The typical shape of this mesh can be seen in Figure 1.

Remark 4.1 If k is small enough, for any two vertices of V^k, there always exists a path given by a natural trajectory of the system that joins the first one with the second one.

Remark 4.2 From (6) and (34) it results that B^k can be generated by

$$B^k = \left\{ x^0 + \sum_{d=1}^{m} \varsigma_d \; e^d \;\; : \;\; \varsigma_d \text{ is an integer} \right\}. \qquad (36)$$

Definition 4.1 Discrete controls associated to the mesh.

We introduce a special family of controls by restricting the distance between switching times in the following way

$$A_x^{d,k} = \left\{ \alpha(\cdot) \in A_x^d \;\; : \;\; \theta_{i+1} - \theta_i = \varsigma\, h^{d_i} \, , \;\; \varsigma \text{ is an integer} \right\} . \tag{37}$$

An Interpretation of the Mesh

The special mesh we use originates a discrete optimal control problem. In that problem, the system has an evolution given by the differential equation (13), but controls d_i are applied during intervals whose length is $\varsigma\, h^{d_i}$ and the initial state x must be a node of V^k.
In consequence, the trajectory associated to this control reaches a node of the mesh at every switching time.
Taking into account the interpretation of the discrete equations as the optimality conditions over the Markov chain associated to this discretization, this interpretation implies that the chain is deterministic in the sense that $P_{i,j} = 0$ or 1, etc. (see [23]).
This property of the mesh plays a key role in relation to the precision of the method and the velocity of convergence of its computational algorithm.

4.1.2 Approximation of the Boundary

We define, $\forall d = 1, \ldots, m$

$$\begin{aligned} \gamma_{k,d}^{-} &= \left\{ x^i \in V^k \;\; : \;\; x^i + h^d\, g(d) \notin Q_k \right\} , \\ \gamma_{k,d}^{=} &= \left\{ x^i \in V^k \;\; : \;\; x^i + h^{\hat{d}}\, g(\hat{d}) \notin Q_k, \;\; \forall \hat{d} \neq d \right\} . \end{aligned} \tag{38}$$

4.1.3 Definition of the approximation space F^k

We consider the set F^k of functions $w : Q^k \times D \to \Re$, $w(\cdot, d) \in W^{1,\infty}(Q_k)$, such that in each quadrilateral element Q_k, $w(\cdot, d)$ is a polynomial which belongs to the Q^1 family (see [6], [10], [11] for the corresponding definitions). It is obvious that any $w \in F^k$ is uniquely characterized by the values $w(x^i, d)$, $x^i \in V^k$, $d \in D$.

4.1.4 Discretization of H-J-B inequalities

We will use the following discretization of conditions (2)

$$\left| \begin{array}{ll} w(x^i, d) \leq D_d^k(w, \mu^k)(x^i) & \forall (x^i, d) \in V^k \times D, \\ & \\ w(x^i, d) \leq S^d(w)(x^i) & \forall (x^i, d) \in V^k \times D. \end{array} \right. \tag{39}$$

We define $D_d^k(w,\mu^k)(x^i)$ in the following form

$$\left|\begin{array}{ll} (D_d^k(w,\mu^k))(x^i) = w(x^i + h^d g(d), d) + \int_0^{h_d} h^d(f(x^i + sg(d), d) - \mu^k)ds & \\ \qquad\qquad \forall x^i \in \left(V^k \bigcap C\gamma_{k,d}^- \bigcap C\left(\bigcup_{r\neq d} \gamma_{k,d}^-\right)\right), & (40) \\ (D_d^k(w,\mu^k))(x^i) = +\infty \qquad \forall x^i \in \left(\gamma_{k,d}^+ \bigcup \left(\bigcup_{r\neq d} \gamma_{k,d}^-\right)\right). & \end{array}\right.$$

Remark 4.3 We can see that the definition of D_d^k is consistent with (2) and takes into account the constraints (23)-(25).

Remark 4.4 We can use (41) instead of (40):

$$\begin{array}{l} (D_d^k(w,\mu^k))(x^i) = w(x^i + h^d g(d), d) + h^d(f(x^i, d) - \mu^k) \\ \qquad \forall x^i \in \left(V^k \bigcap C\gamma_{k,d}^- \bigcap C\left(\bigcup_{r\neq d} \gamma_{k,d}^-\right)\right). \end{array} \tag{41}$$

4.1.5 Definition of operator P^k

We define the operator $P_k : F_k \times \Re \rightarrow F_k$ in the following form

$$P_k(w,\mu^k)(x^i,d) = \min\left((D_d^k(w,\mu^k)), S^d(w)(x^i)\right). \quad \forall (x^i,d) \in V^k \times D. \tag{42}$$

4.1.6 Definition of the discrete problem

In relation to the QVI system, we introduce the following problem which is defined in a similar way to that we employed in [2].

$$\text{Problem } \mathrm{P}_k : \textit{Find } (w,\mu^k) \textit{ such that } w = P_k(w,\mu^k) \tag{43}$$

Definitions of discrete policies and auxiliary concepts

- Set of multi-valued or generalized discrete policies

$$\Lambda = \left\{A : V^k \times D \rightarrow P(D)\right\}.$$

- Set of mono-valued discrete policies

$$\Theta = \left\{A \in \Lambda \ : \ \forall (x^i,d) \in V^k \times D,\ \mathrm{card}(A(x^i,d)) = 1\right\}.$$

- Transitions associated to policies

  Let $A \in \Theta$. We define

$$T(x^i, d) = \left| \begin{array}{ll} T(x^i, d) = x^i & \text{if } d \neq A(x^i, d)\,, \\ \\ T(x^i, d) = x^i + h^d g(d) & \text{if } d = A(x^i, d)\,. \end{array} \right.$$

- Cost of elemental transitions

  Let $w \in F_k$, $x^j \in V^k$ and a policy $A \in \Theta$

$$F_d(x^j, A) = \left| \begin{array}{ll} q(d, A(x^j, d)) & \text{if } d \neq A(x^j, d)\,, \\ \\ h^d\, f_d(A(x^j, d)) & \text{if } d = A(x^j, d)\,. \end{array} \right.$$

- Cycles associated to a policy

  Let a discrete policy $A \in \Theta$, we will say that

$$C = \left\{ (x^1, d_1), ..., (x^q, d_q) \right\}$$

  is one of the cycles associated to that policy if

$$\left| \begin{array}{ll} (x^{j+1}, d_{j+1}) = (T(x^j, d_j), A(x^j, d_j)) & \forall j = 1, ..., q\,, \\ \\ (x^1, d_1) = (T(x^q, d_q), A(x^q, d_q)) & \end{array} \right.$$

- Average costs associated to a policy

  Each cycle C previously defined has the following associated average cost

$$\mu(C) = \frac{\sum\limits_{j=1}^{q} F_d(x^j, A)}{\sum\limits_{j=1,\, d_j \neq 0}^{q} \left(\dfrac{(x^{j+1})_{d_j} - (x^j)_{d_j}}{p_{d_j} - r_{d_j}} \right) + \sum\limits_{j=1,\, d_j = 0}^{q} \left(\dfrac{(x^{j+1})_1 - (x^j)_1}{r_1} \right)}\,.$$

Remark 4.5 For the j index we use the addition modulus q, i.e. $x^{q+1} = x^1$.

Remark 4.6 To unify notation we define

$$\left(\frac{(x^{j+1})_{d_j} - (x^j)_{d_j}}{p_{d_j} - r_{d_j}} \right) = \left(\frac{(x^{j+1})_1 - (x^j)_1}{r_1} \right) \qquad \text{for } d_j = 0,$$

in consequence we obtain the following notation

$$\mu(C) = \frac{\sum_{j=1}^{q} F_d(x^j, A)}{\sum_{j=1}^{q} \left(\frac{(x^{j+1})_{d_j} - (x^j)_{d_j}}{p_{d_j} - r_{d_j}} \right)}.$$

- Optimal discrete control associated to (w, μ) : Φ

$$\Phi(w, \mu) \subset \Lambda$$

$$\Phi(w, \mu)(x^i, d) = \left\{ \tilde{d} \; : \; P_k(w, \mu)(x^i, d) = w(x^i, \tilde{d}) + q(d, \tilde{d}) \right\} \bigcup B(x^i, d),$$

where

$$B(x^i, d) = \left| \begin{array}{ll} \{d\} & \text{if } P_k(w, \mu)(x^i, d) = D_d^k(w, \mu)(x^i, d), \\ \emptyset & \text{if } P_k(w, \mu)(x^i, d) \neq D_d^k(w, \mu)(x^i, d). \end{array} \right.$$

- Family of mono-valued realizations of optimal discrete controls associated to (w, μ), M

$$M(w, \mu) \subset \Theta$$

$$M(w, \mu) = \left\{ A \in \Theta \; : \; A(x^i, d) \in \Phi(w, \mu)(x^i, d), \; \forall (x^i, d) \in V^k \times D \right\}.$$

Proposition 4.1 *There exists at most one value of parameter μ^k such that (43) has a solution $w \in F^k$.*

Proof. Let (w^1, μ_1^k) and (w^2, μ_2^k) be two solutions of (43), such that $\mu_1^k < \mu_2^k$.

For each (w_i, μ_i^k) we can take a policy $A^i \in$M(w_i, μ_i^k).

Let $C = \{(x^1, d_1), ..., (x^q, d_q)\}$ be a cycle associated to A^2 ; for every $j = 1, ..., q$, we have, by virtue of (43)

$$w^1(x^j, d_j) \leq w^1(T(x^j, d_j), A^2(x^j, d_j)) + F_{d_j}(x^j, A^2) - \frac{(x^{j-1})_{d_j} - (x^j)_{d_j}}{p_{d_j} - r_{d_j}} \mu_1^k.$$

In consequence, by addition along the cycle we obtain

$$\sum_{j=1}^{q} w^1(x^j, d_j) \leq \sum_{j=1}^{q} \left(w^1(T(x^j, d_j), A^2(x^j, d_j)) + F_{d_j}(x^j, A^2) - \frac{(x^{j+1})_{d_j} - (x^j)_{d_j}}{p_{d_j} - r_{d_j}} \mu_1^k \right).$$

Taking into account Remark 4.5, we have

$$\sum_{j=1}^{q} w^1(x^j, d_j) \leq \sum_{j=1}^{q} w^1(T(x^j, d_j), A^2(x^j, d_j)),$$

therefore

$$\sum_{j=1}^{q} F_{d_j}(x^j, A^2) \geq \sum_{j=1}^{q} \frac{(x^{j+1})_{d_j} - (x^j)_{d_j}}{p_{d_j} - r_{d_j}} \mu_1^k.$$

In consequence

$$\mu_1^k \leq \frac{\displaystyle\sum_{j=1}^{q} F_{d_j}(x^j, A^2)}{\displaystyle\sum_{j=1}^{q} \frac{(x^{j+1})_{d_j} - (x^j)_{d_j}}{p_{d_j} - r_{d_j}}}.$$

In the same way we obtain

$$\mu_2^k = \frac{\displaystyle\sum_{j=1}^{q} F_{d_j}(x^j, A^2)}{\displaystyle\sum_{j=1}^{q} \frac{(x^{j+1})_{d_j} - (x^j)_{d_j}}{p_{d_j} - r_{d_j}}}.$$

Therefore it results $\mu_2^k \geq \mu_1^k$, which is a contradiction. Then $\mu_2^k = \mu_1^k$.

□

Remark 4.7 If (w, μ) is a solution of (43), then $w + c \cdot e$ is a solution $\forall c \in \Re$, being $e = (1, ..., 1) \in \Re^{(m-1)\times N}$.

4.1.7 Definition of operator P_k^λ

We repeat here the definition of operator $D_d^{\lambda,k}$, introduced in [2]

$$\left|\begin{array}{ll} (D_d^{\lambda k} w)(x^i) = (1 - \lambda\, h^d)\, w(x^i + h^d g(d), d) + h^d f(x^i, d) & \\ \qquad\qquad \forall x^i \in \left(V^k \bigcap C\gamma_{k,d}^- \bigcap C\left(\bigcup_{r\neq d} \gamma_{k,d}^-\right)\right). & \\ & \\ (D_d^k w)(x^i) = +\infty \qquad \forall x^i \in \left(\gamma_{k,d}^+ \bigcup \left(\bigcup_{r\neq d} \gamma_{k,d}^-\right)\right). & \end{array}\right. \tag{44}$$

The operator $P_k^\lambda : F_k \to F_k$ is defined by

$$(P_k\, w)(x^i, d) = \min\left((D_d^k w)(x^i),\ (S^d(w))(x^i)\right), \qquad \forall x^i \in V^k,\ \forall d \in D, \tag{45}$$

and the following problem allows us to find the unique solution $U^{\lambda,k}$ of the discrete discounted cost problem

Problem P_k^λ : *Find the fixed point of operator* P_k^λ.

4.2 Relation between problems $\mathbf{P}_k$ and $\mathbf{P}_k^\lambda$

The relation between these two problems is established in the following form.

Theorema 4.1 *By virtue of feasibility condition* (6), *we have*

$$\lim_{\lambda\to 0} \lambda\, U^{\lambda,k}(x^i,d) = \mu^k \qquad \forall x^i \in V^k,\ \forall d \in D. \tag{46}$$

Also, $\forall x^{io} \in V^k$, $\forall d_0 \in D$

$$\forall w \in \left(\bigcap_{\varsigma>0}\left(\overline{\bigcup_{\varsigma>\lambda>0} \left(U^{\lambda,k}(\cdot) - U^{\lambda,k}(x^{io},d_0)\cdot e\right)}\right)\right), \qquad (w,\mu^k) \text{ is a solution of } P_k. \tag{47}$$

Proof. In first place let us see that for each pair (x^i,d), $\{\lambda\, U_d^{\lambda,k}(x^i,d) \,:\, \lambda \in \Re^-\}$ is bounded.
Let $(x^i,d) \in V^k \times D$, by virtue of the construction of the mesh, there exists a closed cycle to which it belongs; therefore, there exists $M(k)$ (which does not depend on λ) such that

$$U^{\lambda,k}(x^i,d) \le \frac{M(k)}{\lambda}, \tag{48}$$

then

$$0 \le \lambda\, U^{\lambda,k} \le M(k).$$

Moreover, two points (x^i,d) and (x^{io},d_0) of $V^k \times D$, can be joined by a path. This path has an associated cost that is uniformly-bounded with respect to λ. in consequence there exists $K(x^i,x^{io},d,d_0)$ such that

$$\left|U^{\lambda,k}(x^i,d) - U^{\lambda,k}(x^{io},d_0)\right| \le K(x^i,x^{io},d,d_0). \tag{49}$$

Let $\{\lambda^\nu\}$ be a sub-sequence such that there exists $\tilde\mu$, $w(x^i,d)$ for which the following convergence holds

$$\lambda^\nu\, U^{\lambda^\nu,k}(x^{io},d_0) \to \tilde\mu, \tag{50}$$

$$U^{\lambda,k}(x^i,d) - U^{\lambda,k}(x^{io},d_0) \to w(x^i,d). \tag{51}$$

By virtue of (49), $\forall x^i$ and $\forall d$

$$\lambda^\nu\, U^{\lambda^\nu,k}(x^{io},d_0) \to \tilde\mu\,. \tag{52}$$

Let us see that $(w,\tilde\mu)$ is a solution of P^k. We have, $\forall x^i \in V^k$, $\forall d = 0,\dots,m$,

$$U^{\lambda^\nu,k}(x^i,d) - U^{\lambda^\nu,k}(x^{io},d_0) = (P_k^\lambda U^{\lambda^\nu,k})(x^i,d) - U^{\lambda^\nu,k}(x^{io},d_0) =$$

$$\min\left(\left(D_d^{\lambda,k}U^{\lambda^\nu,k}\right)(x^i) - U^{\lambda^\nu,k}(x^{io},d_0)\,,\ \left(S^d\, U^{\lambda^\nu,k}\right)(x^i) - U^{\lambda^\nu,k}(x^{io},d_0)\right).$$

$$\left(D_d^{\lambda,k}U^{\lambda^\nu,k}\right)(x^i) - U^{\lambda^\nu,k}(x^{io},d_0) =$$

$$U^{\lambda^\nu,k}\left(x^i + h^d g(d), d\right) - U^{\lambda^\nu,k}\left(x^{io},d_0\right) + h^d f(x^i,d) - \lambda^\nu h^d U^{\lambda^\nu,k}\left(x^i + h^d g(d), d\right).$$

By taking limit when $\lambda^\nu \to 0$, we obtain

$$\left(D_d^k(w,\tilde{\mu})\right)(x^i) = w\left(x^i + h^d g(d), d\right) + h^d \cdot \left(f(x^i,d) - \tilde{\mu}\right).$$

Moreover, as

$$\left(S^d U^{\lambda^\nu,k}\right)(x^i) - U^{\lambda^\nu,k}(x^{io},d_0) = \min_{\tilde{d}\neq d}\left(q(d,\tilde{d}) + U^{\lambda^\nu,k}(x^i,\tilde{d}) - U^{\lambda^\nu,k}(x^{io},d_0)\right),$$

we have

$$\lim_{\lambda^\nu\to 0}\left(S^d U^{\lambda^\nu,k}\right)(x^i) - U^{\lambda^\nu,k}\left(x^{io},d_0\right) = S^d(w)(x^i).$$

Finally, $\forall x^i \in V^k$, $\forall d = 0,\dots,m$, we have

$$\lim_{\lambda^\nu\to 0} U^{\lambda^\nu,k}(x^i,d) - U^{\lambda^\nu,k}(x^{io},d_0)$$

$$= \lim_{\lambda^\nu\to 0}\left(\min\left(\left(D_d^{\lambda,k}U^{\lambda^\nu,k}\right)(x^i) - U^{\lambda^\nu,k}\left(x^{io},d_0\right), \left(S^d U^{\lambda^\nu,k}\right)(x^i) - U^{\lambda^\nu,k}\left(x^{io},d_0\right)\right)\right)$$

$$= \min\left(\lim_{\lambda^\nu\to 0}\left(D_d^{\lambda,k}U^{\lambda^\nu,k}\right)(x^i) - U^{\lambda^\nu,k}\left(x^{io},d_0\right), \lim_{\lambda^\nu\to 0}\left(S^d U^{\lambda^\nu k}\right)(x^i) - U^{\lambda^\nu,k}\left(x^{io},d_0\right)\right)$$

$$= \min\left(\left(D_d^k(w,\tilde{\mu})\right)(x^i),\ S^d(w)(x^i)\right),$$

i.e., we have

$$w(x^i,d) = \min\left(\left(D_d^k(w,\tilde{\mu})\right)(x^i),\ S^d(w)(x^i)\right),$$

in consequence $(w,\ \tilde{\mu})$ is a solution of P^k. From Proposition 4.1 it results

$$\tilde{\mu} = \mu^k.$$

As the subsequence λ^ν that we have chosen is arbitrary, we conclude that there exists a unique accumulation point. This conclusion implies that all the sequence is convergent, i.e.

$$\lim_{\lambda\to 0} \lambda\, U^{\lambda,k}(x^i,d) = \mu^k.$$

□

Remark 4.8 In fact it can be proved – using basically the fact that the set of feedback policies of the discrete problem is finite – that the following estimation of the convergence velocity holds .

$$\left|\lambda\, U^{\lambda,k} - \mu^k\right| \le C(k)\,\lambda.$$

The proof uses elementary arguments and it will be omitted for the sake of briefness.

4.3 Convergence of the method

Theorema 4.2 *The following estimation for the difference between the optimal average cost of the original continuous problem and those corresponding to its discrete approximation holds*

$$\left|\mu - \mu^k\right| \le C\,k. \tag{53}$$

Proof. We use the following triangular inequality

$$\left|\mu - \mu^k\right| \le \left|\mu - \lambda U^\lambda\right| + \lambda \left\|U^\lambda - U^{\lambda,k}\right\| + \left|\lambda U^{\lambda,k} - \mu^k\right|.$$

By taking limit when $\lambda \to 0$ we obtain (53), since by virtue of [2] it is verified

$$\left\|U^\lambda - U^{\lambda,k}\right\|_K \le \frac{C(K)}{\lambda}\,k,$$

being

$$\left\|U^\lambda - U^{\lambda,k}\right\|_K = \max\left\{\left|U_d^\lambda(x) - U^{\lambda,k}(x,d)\right| \;:\; x \in V^k \bigcap K,\ d \in D\right\},$$

where K is a compact set that does not depend on k such that

$$K \subset Q,\ K \cap \partial Q^- = \emptyset.$$

□

5 Numerical algorithm

We define here an algorithm which uses value iteration and policy iteration techniques and also makes use of the properties established in Theorem 3.2. This algorithm takes into account the methods described in [17], [21] and it is a natural modification of the algorithm presented in [2].

5.1 Preliminary definitions

- ϵ–suboptimal multi-valued discrete controls associated to w: A_ϵ

$$\begin{aligned} & A_\epsilon \in \Lambda \\ & (A_\epsilon\, w)(x^i,d) = (B_\epsilon\, w)(x^i,d) \bigcap (C_\epsilon\, w)(x^i,d)\,, \end{aligned} \tag{54}$$

  where

$$(B_\epsilon\, w)(x^i,d) = \left\{\widetilde{d} \in D \;:\; (P_k w)(x^i,d) + \epsilon \ge q(d,\widetilde{d}) + w(x^i,\widetilde{d})\right\},$$

$$(C_\epsilon\, w)(x^i,d) = \left|\begin{array}{lll} \{d\} & \text{if} & (P_k w)(x^i,d) \ge (D_d^k\, w)(x^i) - \epsilon\,, \\ \emptyset & \text{if} & (P_k w)(x^i,d) < (D_d^k\, w)(x^i) - \epsilon\,. \end{array}\right. \tag{55}$$

- Linear system associated to a feedback discrete policy $\tilde{A} \in \Theta$.

We consider a linear system that we denote in a brief way

$$L(u, \mu) = b\,. \tag{56}$$

For this system, the relation that defines each equation is

$$\left|\begin{array}{ll} u(x^i, d) = (D_d^k\, u, \mu)(x^i) & \text{if } \tilde{A}(u)(x^i, d) = d\,, \\ \\ u(x^i, d) = q(d, \widetilde{d}) + u(x^i, \widetilde{d}) & \text{if } \tilde{A}(u)(x^i, d) = \tilde{d}\,. \end{array}\right.$$

For the discount problem λ, we also consider the linear system associated to a policy $\tilde{A} \in \Theta$

$$L^\lambda u = \beta\,, \tag{57}$$

where the relation that defines each equation is

$$\left|\begin{array}{ll} u(x^i, d) = (D_d^{\lambda,k}\, u)(x^i) & \text{if } \tilde{A}(u)(x^i, d) = d\,, \\ \\ u(x^i, d) = q(d, \widetilde{d}) + u(x^i, \widetilde{d}) & \text{if } \tilde{A}(u)(x^i, d) = \tilde{d}\,. \end{array}\right.$$

5.2 A fast algorithm

Step 1: Set $\lambda > 0$, $0 < \gamma < 1$, $w^{1,0} \in F_k$, $\bar{r} > 0$, $\epsilon > 0$, $\nu = 0$, $\lambda^\nu = \lambda$

Step 2: $\nu = \nu + 1$, $\eta = 0$, $A^\nu = \{\emptyset\}^{(m+1)\times N}$

Step 3: $\eta = \eta + 1$, compute $w^{\nu,\eta} = P_k^{\lambda^\nu}(w^{\nu,\eta-1})$, $A^{\nu,\eta} = A_\epsilon(w^{\nu,\eta})$

Step 4: If $A^{\nu,\eta} = A^\nu$, $r = r + 1$, and go to Step 5
else, $r = 0$, $A^\nu = A^{\nu,\eta}$, and go to Step 3.

Step 5: If $r \geq \bar{r}$, go to Step 3
else, choose any $\hat{A}^{\nu,\eta} \in \Theta$ such that
$\forall (x^i, d)$, $\hat{A}^{\nu,\eta}(x^i, d) \in A^{\nu,\eta}(x^i, d)$
and construct the system L^λ associated to $\hat{A}^{\nu,\eta}$

Step 6: If $\det(L^\lambda) \neq 0$, compute the solution v of the system $L^\lambda v = \beta$
else, $r = 0$, $A^\nu = \{\emptyset\}^{(m+1)\times N}$, and go to Step 3.

Step 7: If $v \neq P_k^{\lambda^\nu} v$, if ($\nu = 0$ or $v \leq w^{\nu,\eta}$), $w^{\nu+1,0} = v$, $\nu = \nu + 1, \eta = 0$, $r = 0$,
$A^\nu = \{\emptyset\}^{(m-1)\times N}$, and go to Step 3

Step 8: Form the system L associated to $\hat{A}^{\nu,\eta}$

Step 9: Test if the system $L(u, \mu^k)$ has at least one solution and compute it
else, go to Step 11

Step 10: If $\mathbf{TEST}(u, \mu^k, \hat{A}^{\nu,\eta}) = 1$, end ($\hat{A}^{\nu,\eta}$ is an optimal policy)

Step 11: $w^{\nu-1,0} = \dfrac{1}{\gamma} v$, $\lambda^{\nu+1} = \gamma\, \lambda^\nu$ and go to Step 2

Remark 5.1 Essentially the function **TEST** must check that the policy $\hat{A}^{\nu,\eta}$, is optimal for the discrete problem with average criterion. This property holds if the pair (u, μ^k) is a solution of (43).

Although the pair could not be a solution of (43), the policy $\hat{A}^{\nu,\eta}$ may be optimal in the sense that there is another pair (w, μ^k) which is a solution of the system $L(w, \mu^k) = b$, and that in addition it is a solution of (43). If this holds, **TEST** brings a function with that property.

5.2.1 Definition of the auxiliary function TEST

The function $\mathbf{TEST}(u, \mu^k, \hat{A})$ is computed by the following algorithm:

Step 1: $v = 1,\ w^1 = u,\ A^r = \hat{A}$
compute $S^{\tilde{c}} = \left\{ \text{cycles associated to } \hat{A} \right\}$

Step 2: if $w^v = P_k(w^v, \mu^k)$ **TEST**=1. End.

Step 3: $v = P_k(w^v, \mu^k)$
choose (x^0, d_0), such that $v(x^0, d_0) < w^v(x^0, d_0)$

Step 4: compute $\Theta(w^v, \mu^k)$ and choose any $\tilde{A} \in \mathrm{M}(w^v, \mu^k)$
form $A \in \Theta$ in the following way:

$$A(x^i, d) = A^r(x^i, d), \text{ if } (x^i, d) \neq (x^0, d_0)$$
$$A(x^0, d_0) = \tilde{A}(x^0, d_0)$$

Step 5: compute $C = \{\text{circles associated to } A\}$
if C is not included in $\tilde{C}$, **TEST** $= 0$. End
else, find the maximum element of the set of vectors w that

$$\left| \begin{array}{l} L(w, \mu^k) = b, \\ \\ \max\limits_{C} w \leq 0, \end{array} \right.$$

where L is the linear system associated to A
$v = v + 1,\ w^v = w,\ A^r = A$, and go to Step 2.

Remark 5.2 The function **TEST** is defined in such a way that when for a pair w – solution of the L system associated to policy $\hat{A}^{\nu,\eta}$ – the optimality condition is satisfied, we have

$$\mathbf{TEST}(u, \mu^k, \hat{A}^{\nu,\eta}) = 1.$$

Remark 5.3 It is clear that the maximum element computed in Step 5, can be found by solving a linear programming problem. The existence of maximum element is immediate and the search of this element can be solved by an ad-hoc algorithm without using a general linear programming procedure.

Proposition 5.1 *The algorithm, which computes the values of the* **TEST** *function, converges in a finite number of steps.*

Proof. Let us suppose, by *reductio ad absurdum*, that the algorithm originates an infinite sequence of values w^v. By construction, it would generate a sequence of Markov's chains where the sets of associated cycles are non-increasing in the following sense

$$C(A^{v+1}) \subseteq C(A^v).$$

In consequence, after a finite number of steps $C(A^{\bar{v}})$ would remain fixed. From this condition it is possible to see that the sequence of functions w^v is decreasing and can only take a finite number of values. This is a contradiction to the supposition that the sequence is infinite.

□

Lemma 5.1 *If in the previous algorithm the set $S^{\hat{c}}$ is not included in $S^{\bar{c}}$, there exists another cycle C such that $\mu(C) < \mu^k$.*

Proof. Let C be a new cycle with q nodes that, without losing generality, we will denote by

$$C = \left\{(x^1, d_1), ..., x^q, d_q)\right\},$$

i.e.

$$\left| \begin{array}{l} (x^{j+1}, d_{j+1}) = (T(x^j, d_j),\ A(x^j, d_j)), \\ (x^1, d_1) = (T(x^q, d_q),\ A(x^q, d_q)). \end{array} \right.$$

By construction, we have that for every $j = 1, ..., q$ it is verified

$$w(x^j, d_j) \geq w(T(x^j, d_j), A(x^j, d_j)) + F_{d_j}(x^j, x^{j-1}, A(x^j, d_j)) - \left(\frac{(x^{j-1})_{d_j} - (x^j)_{d_j}}{p_{d_j} - r_{d_j}}\right)\mu^k,$$

where at least one inequality is strict, hence

$$\sum_{j=1}^{q} w(x^j, d_j) \geq$$

$$\sum_{j=1}^{q}\left(w(T(x^j, d_j), A(x^j, d_j)) + F_{d_j}(x^j, x^{j-1}, A(x^j, d_j)) - \left(\frac{(x^{j-1})_{d_j} - (x^j)_{d_j}}{p_{d_j} - r_{d_j}}\right)\mu^k\right).$$

As

$$\sum_{j=1}^{q} w(x^j, d_j) = \sum_{j=1}^{q} w(T(x^j, d_j), A(x^j, d_j)),$$

we obtain

$$\sum_{j=1}^{q} F_{d_j}(x^j, x^{j-1}, A(x^j, d_j)) < \sum_{j=1}^{q}\left(\frac{(x^{j-1})_{d_j} - (x^j)_{d_j}}{p_{d_j} - r_{d_j}}\right)\mu^k.$$

In consequence, the average $\tilde{\mu}^k$ along the new cycle satisfies

$$\mu^k > \mu(C) = \frac{\sum_{j=1}^{q} F_{d_j}(x^j, x^{j+1}, A(x^j, d_j))}{\sum_{j=1}^{q} \left(\frac{(x^{j+1})_{d_j} - (x^j)_{d_j}}{p_{d_j} - r_{d_j}} \right)}.$$

□

5.3 Convergence of the algorithm

Theorema 5.1 *The algorithm converges in a finite number of steps*

Proof. The algorithm cannot remain in an infinite loop from Step 3 to Step 7 by virtue of [13]. We will also see that the algorithm can not produce an infinite sequence by passing through Step 11. In effect, if it passed through Step 11 an infinite number of times, it would generate a sequence $\lambda^\nu \to 0$.
In fact, for the $\tilde{A}^{\nu,\eta}$ obtained after Step 7, we have that for every cycle associated to C,

$$\lambda^\nu U^{\lambda^\nu,k}(x^i, d) \to \mu(C) \qquad \forall (x^i, d) \in C.$$

At the same time, by virtue of Theorem 4.1

$$\lambda^\nu U^{\lambda^\nu,k}(x^i, d) \to \mu^k \qquad \forall (x^i, d) \in V^k \times D,$$

then, from some $\bar{\nu}$ it results $\forall C$

$$\mu(C) = \mu^k,$$

i.e. the policy $\tilde{A}^{\nu,\eta}$, is optimal, then it results **TEST**=1 and the algorithm finishes in a finite number of steps generating the optimal discrete solution.

□

6 Applications

We have applied the above presented procedure to an example with $m = 2$ items and a discretization of Q which comprises 50×50 nodes. The instantaneous cost function is linear and does not depend on the parameter d, i.e.

$$f(x_1, x_2) = C_1\, x_1 + C_2\, x_2 \qquad \forall d \in D.$$

In Figure 2 it is shown the optimal trajectory obtained.

$M_1 = 0.833$	$M_2 = 0.833$
$r_1 = 1$	$r_2 = 1$
$p_1 = 6$	$p_2 = 1.5$
$C_1 = 0.1$	$C_2 = 0.1$
$q_{0,1} = 15$	$q_{1,0} = 0$
$q_{0,2} = 3$	$q_{2,0} = 0$
$q_{2,1} = 15$	$q_{1,2} = 3$
$h_1 = 0.017$	

Table 1:

7 Conclusions

In this paper, we have presented both analytic and numerical results concerning the optimization of the production schedule of a multi-item single machine system, for the case of infinite horizon and average cost criterion.

We have analyzed the associated HJB equation (which here takes the form of a variational inequalities system of H-J-B type). We have proved the existence of solution of this QVI system and the uniqueness of the optimal average cost.

We have also obtained a method of discretization which principal features is that the approximation has a k-order precision. This property stems from the fact that the optimal cost functions of the problems with actualization coefficients λ are uniformly Lipschitz continuous with respect to the parameter λ, and from the especial type of mesh used in the triangulation procedure.

In addition, we have developed a computational algorithm that obtains the solution of the discrete problem in a finite number of steps. This property is also a consequence of the especial type of mesh used.

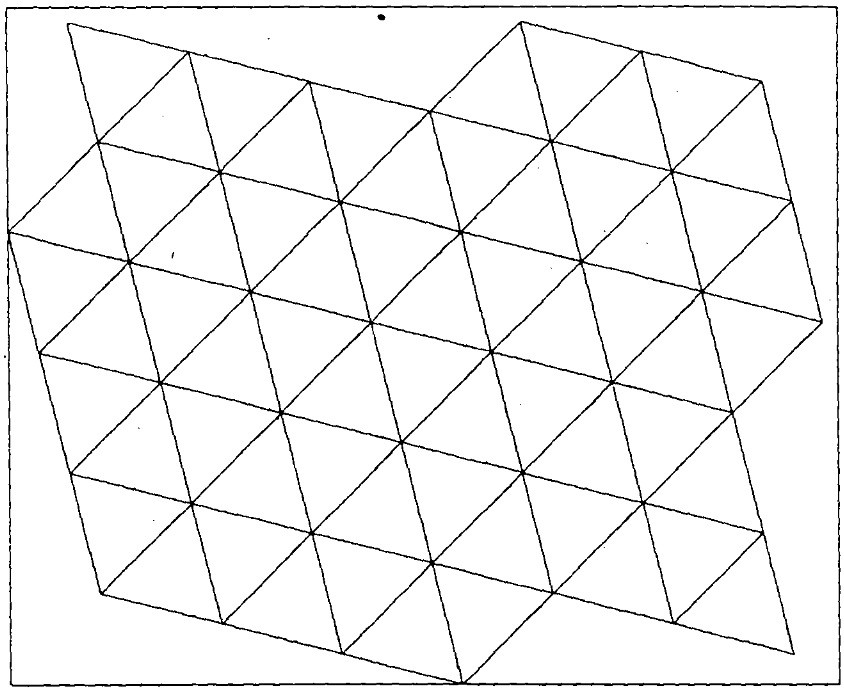

Figure 1: The mesh of Ω

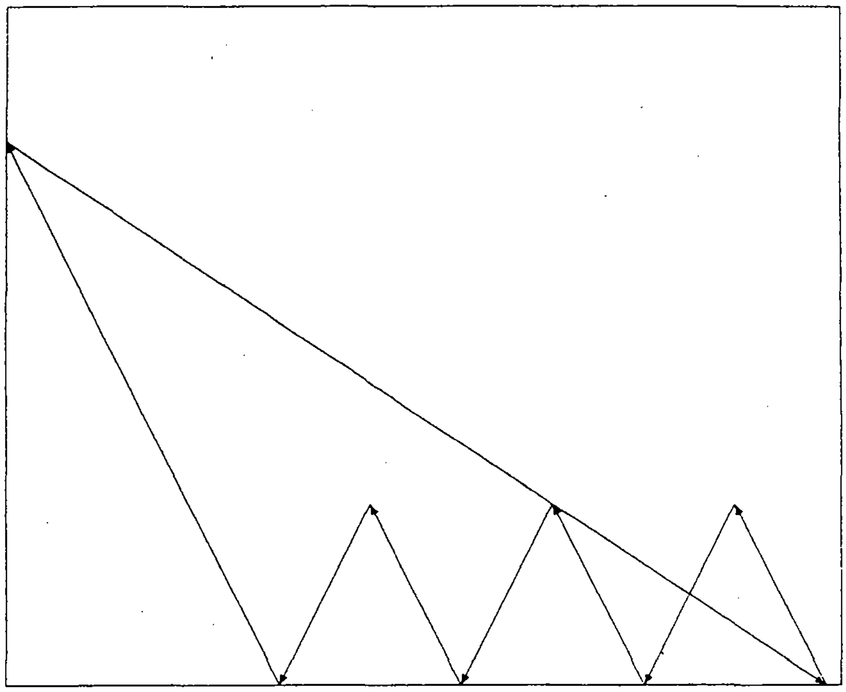

Figure 2: State space trajectory

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Éditeur
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ISSN 0249 - 6399

RR- 2984